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Constraint-induced mean curvature dependence of Cartesian momentum operators

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Abstract

The Hermitian Cartesian quantum momentum operator **p** for an embedded surface M in R^3 is proved to be a constant factor $-\mathrm{i}\hbar$ times the mean curvature vector field $H\mathbf{n}$ added to the usual differential term. With use of this form of momentum operators, the operator-ordering ambiguity exists in the construction of the correct kinetic energy operator and three different operator-orderings lead to the same result.

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1. Introduction

For a particle moving on the curved smooth (regular) surface M embedded in R^3, which is parameterized by two local coordinates (ξ, ζ), the quantum kinetic energy operator takes the following form,

$$T \equiv -\frac{\hbar^2}{2m}\nabla^2 = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{g}}\partial_\mu g^{\mu\upsilon}\sqrt{g}\partial_\upsilon\,, \tag{1}$$

where

$$\nabla^2 = \partial_i\partial_i = \frac{1}{\sqrt{g}}\partial_\mu g^{\mu\upsilon}\sqrt{g}\partial_\upsilon \tag{2}$$

is the Laplace–Beltrami operator [1]. The symbol ∂ stands for differential operator as usual. The metric tensor $g_{\mu\upsilon}$ is defined via the length element square $\mathrm{d}s^2 = g_{\mu\upsilon}\,\mathrm{d}x^\mu\,\mathrm{d}x^\upsilon$ and $\mathrm{d}\sigma = \sqrt{g}\,\mathrm{d}\xi\,\mathrm{d}\zeta$ is the area element on the surface. The factor $g \equiv \det(g_{\mu\upsilon})$ is the determinant of the matrix formed by the metric tensor. In this paper the Latin indices (i, j, k) are used to denote the Cartesian coordinates (x, y, z) with $x^i = x_i$ and Greek indices (μ, υ) to denote the local ones (ξ, ζ) with $x^\mu = g^{\mu\upsilon}x_\upsilon$. The convention the repeated indices mean summation is implied unless specified. Only two-dimensional surface embedded in the three-dimensional Euclidean space is addressed in this paper because in majority of the realistic constraint problems, the motion is on the two-dimensional curved surfaces [2, 3]. However, our conclusion can be readily generalized to the higher-dimensional manifold.

For the constraint motion, the quantum kinetic energy operator can be rewritten into a form depending on the *generalized momentum operators* p_μ as [1]

$$T \equiv \frac{1}{2m}\frac{1}{g^{1/4}}p_\mu g^{1/4}g^{\mu\upsilon}g^{1/4}p_\upsilon\frac{1}{g^{1/4}} \tag{3}$$

where the generalized momentum operators p_μ ($\mu = \xi, \zeta$) are with $\Gamma_\mu \equiv \Gamma^\upsilon_{\mu\upsilon}$ being the once-contracted affine connection

$$p_\mu = -\mathrm{i}\hbar(\partial_\mu + 1/2\Gamma_\mu). \tag{4}$$

In the kinetic energy (3), the four identical $g^{1/4}$ factors are used to fix the operator-ordering problem, and they are so inserted that the standard result (1) can be restored. In classical limit, these factors drop out and equation (3) becomes

$$T \equiv \frac{1}{2m}g^{\mu\upsilon}p_\mu p_\upsilon. \tag{5}$$

Similarly, when examining the same constraint motion in Cartesian coordinates with use of the Hermitian form of *Cartesian momentum* p_i ($i = x, y, z$), the elaboration of the kinetic energy operator should also take appropriate account of the operator-ordering problem. In analogy of (3) the quantum kinetic energy operator may take the following form

$$T = \frac{1}{2m}\sum_{i=1}^{3}\sum_{i=1}^{3}\frac{1}{f_i}p_i f_i^2 p_i\frac{1}{f_i}, \tag{6}$$

where the Cartesian momentum p_i depends on two independent curved coordinates (ξ, ζ) and their first derivatives only, and the operator-ordering factors f_i ($i = x, y, z$) are non-trivial functions depending on the local coordinates (ξ, ζ) too. When the constraint is removed or the motion is in classical limit, the factors $f_i(x, y, z)$ cancel out; and the kinetic energy operator (6) reduces to be its usual form

$$T = \frac{1}{2m}p_i p_i. \tag{7}$$

It can be anticipated that the Hermitian form of the Cartesian quantum momentum operators p_i may take a form similar to (4), which proves to be

$$p_i = -\mathrm{i}\hbar(\partial_i + Hn_i), \tag{8}$$

where H is the *mean curvature* of the surface M in which $\mathbf{n} = (n_x, n_y, n_z)$ denoting the unit normal vector on the surface, and the quantity $H\mathbf{n}$ is an existing *geometric invariant* in differential geometry, the so-called mean curvature vector field [4].

This paper is organized as what follows. A proof of result (8) is given in section 2. The condition for the operator-ordering factors f_i being able to convert equation (6) into equation (1) is derived in section 3, which is found to depend on the mean curvature H too. However, the way of inserting f_i into $p_i p_i$ (7) is not unique, and two other ways of the insertion can yield the correct result (1), as shown in section 4. To illustrate the abstract formulae obtained, the explicit results for two surfaces are given in section 5. In final section 6, some remarks are provided.

2. The Hermitian Cartesian quantum momentum operator

The standard representation of the curved smooth surface M embedded in R^3 is

$$\mathbf{r}(\xi, \zeta) = (x(\xi, \zeta), y(\xi, \zeta), z(\xi, \zeta)). \tag{9}$$

The covariant derivatives of $\mathbf{r}$ (9) are $\mathbf{r}_\mu = \partial \mathbf{r}/\partial x^\mu$, and then the metric tensor $g_{\mu\upsilon}$ is easily formed as $g_{\mu\upsilon} \equiv \mathbf{r}_\mu \cdot \mathbf{r}_\upsilon$. The normal vector at point (ξ, ζ) is $\mathbf{n} = \mathbf{r}^\xi \times \mathbf{r}^\zeta/\sqrt{g}$. The contravariant derivatives $\mathbf{r}^\mu \equiv g^{\mu\upsilon}\mathbf{r}_\upsilon$ are the generalized inverse (or pseudoinverse, or Moore–Penrose inverse) of the covariant ones $\mathbf{r}_\mu$ for we have $\mathbf{r}^\mu \cdot \mathbf{r}_\upsilon = g^{\mu\alpha}\mathbf{r}_\alpha \cdot \mathbf{r}_\upsilon = g^{\mu\alpha}g_{\alpha\upsilon} = \delta^\mu_\upsilon$. The derivatives $\mathbf{r}^\mu$ and $\mathbf{r}_\upsilon$ actually constitute the transformation matrix between ∂_i and ∂_μ, and explicitly we have

$$\partial_i = x_i^\mu \partial_\mu, \qquad \text{and} \qquad \partial_\mu = x_\mu^i \partial_i. \tag{10}$$

In consequence, operator $\partial_i\partial_i = x_i^\mu \partial_\mu x_i^\upsilon \partial_\upsilon$ is the Laplace–Beltrami operator (2) for the surface,

$$\partial_i\partial_i = x_i^\mu \partial_\mu x_i^\upsilon \partial_\upsilon = \mathbf{r}^\mu \partial_\mu \cdot \mathbf{r}^\upsilon \partial_\upsilon = g^{\mu\upsilon}\partial_\mu\partial_\upsilon - \Gamma^{\mu\upsilon}_\mu \partial_\upsilon = \nabla^2, \tag{11}$$

where the Gauss formula $\partial_\mu \mathbf{r}^\upsilon = -\Gamma^\upsilon_{\gamma\mu}\mathbf{r}^\gamma + b^\upsilon_\mu \mathbf{n}$ [4] is used. Using the Bohm's rule [5], we obtain the Hermitian form of the operators $-\mathrm{i}\hbar\partial_i$, and it is

$$\begin{aligned} p_i &\equiv \frac{1}{2}\{(-\mathrm{i}\hbar\partial_i + (-\mathrm{i}\hbar\partial_i)^\dagger\} \\ &= -\mathrm{i}\hbar\left\{x_i^\mu\partial_\mu + \frac{1}{2\sqrt{g}}\partial_\mu\left(\sqrt{g}x_i^\mu\right)\right\} \\ &= -\mathrm{i}\hbar\left\{x_i^\mu\partial_\mu + H_i\right\}, \qquad (i = 1, 2, 3), \end{aligned} \tag{12}$$

where

$$H_i \equiv \frac{1}{2\sqrt{g}}\partial_\mu\left(\sqrt{g}x_i^\mu\right) \tag{13}$$

is the constraint induced term. Rewriting (13) into the vector form, we see

$$\mathbf{H} \equiv \frac{1}{2\sqrt{g}}\partial_\mu(\sqrt{g}\mathbf{r}^\mu) = \frac{1}{2\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\upsilon}\partial_\upsilon\mathbf{r}) = \frac{1}{2}\nabla^2\mathbf{r} = H\mathbf{n}. \tag{14}$$

In the last step, the formula $\nabla^2\mathbf{r} = 2H\mathbf{n}$ [6] is used. For those who are unfamiliar with this formula, another straightforward proof is available. Recalling the Gauss formula $\partial_\upsilon \mathbf{r}^\mu = -\Gamma^\mu_{\gamma\upsilon}\mathbf{r}^\gamma + b^\mu_\upsilon\mathbf{n}$ [4] and using two relations $\Gamma^\upsilon_{\mu\upsilon} = \partial_\mu \ln\sqrt{g}$ and $b^\mu_\mu \equiv g^{\mu\upsilon}b_{\mu\upsilon} = 2H$ [4], we have, for $\mathbf{H}$,

$$\mathbf{H} = \tfrac{1}{2}\left(\partial_\mu\mathbf{r}^\mu + \Gamma^\upsilon_{\mu\upsilon}\mathbf{r}^\mu\right) = \tfrac{1}{2}\left(-\Gamma^\upsilon_{\mu\upsilon}\mathbf{r}^\mu + b^\mu_\mu\mathbf{n} + \Gamma^\upsilon_{\mu\upsilon}\mathbf{r}^\mu\right) = H\mathbf{n}. \tag{15}$$

Thus, the Hermitian Cartesian momentum $\mathbf{p}$ (12) is in its final form

$$\mathbf{p} = -\mathrm{i}\hbar(\mathbf{r}^\mu\partial_\mu + H\mathbf{n}). \tag{16}$$

When the motion is constraint-free or in a flat plane, i.e., when $H = 0$, the constraint-induced terms $H\mathbf{n}$ vanish. Then the Cartesian momentum operator (16) reproduces its usual form as

$$\mathbf{p} = -\mathrm{i}\hbar\nabla. \tag{17}$$

3. Kinetic operator in terms of the Hermitian Cartesian momentum operators

With use of the Hermitian form of momentum operator (16), the correct kinetic energy operator can no longer be expressed by

$$T = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right), \tag{18}$$

which will be shortly seen to include an excess positive term $(\hbar^2/2m)H^2$ in comparison with the correct kinetic operator (1),

$$\frac{1}{2m}\left(p_x^2+p_y^2+p_z^2\right)=-\frac{\hbar^2}{2m}\nabla^2+\frac{\hbar^2}{2m}H^2. \tag{19}$$

So, the operator-ordering problem must be taken into consideration, and we can resort to the form of equation (6). Substituting p_i (16) into equation (6), we have

$$\begin{aligned}T &= \frac{1}{2m}\sum_{i=1}^{3}\frac{1}{f_i(x,y,z)}p_i f_i(x,y,z) f_i(x,y,z) p_i \frac{1}{f_i(x,y,z)}\\ &= -\frac{\hbar^2}{2m}\sum_{i=1}^{3}\left(\frac{1}{f_i}x_i^{\mu}\partial_\mu f_i + H_i\right)\left(f_i x_i^{\upsilon}\partial_\upsilon\frac{1}{f_i}+H_i\right)\\ &= -\frac{\hbar^2}{2m}\sum_{i=1}^{3}\left(x_i^{\mu}\partial_\mu + H_i + x_i^{\mu}(\partial_\mu \ln f_i)\right)\left(x_i^{\upsilon}\partial_\upsilon + H_i - x_i^{\upsilon}(\partial_\upsilon \ln f_i)\right)\\ &= -\frac{\hbar^2}{2m}\left(x_i^{\mu}\partial_\mu + H_i + R_i\right)\left(x_i^{\upsilon}\partial_\upsilon + H_i - R_i\right),\end{aligned} \tag{20}$$

where

$$R_i \equiv x_i^{\mu}(\partial_\mu \ln f_i), \qquad \text{(no summation over two repeated indices } i). \tag{21}$$

Expanding the right-hand side of equation (20), we find

$$\begin{aligned}T &= -\frac{\hbar^2}{2m}\left(x_i^{\mu}\partial_\mu x_i^{\upsilon}\partial_\upsilon + x_i^{\mu}\partial_\mu H_i - x_i^{\mu}\partial_\mu R_i + H_i x_i^{\upsilon}\partial_\upsilon + R_i x_i^{\upsilon}\partial_\upsilon + H_i^2 - R_i^2\right)\\ &= -\frac{\hbar^2}{2m}\left(x_i^{\mu}\partial_\mu x_i^{\upsilon}\partial_\upsilon + \left\{2H_i x_i^{\mu}\right\}\partial_\mu + \left\{x_i^{\mu}((\partial_\mu H_i)-(\partial_\mu R_i)) + H_i H_i - R_i R_i\right\}\right).\end{aligned} \tag{22}$$

Because of $\mathbf{H} = H\mathbf{n}$ and $\mathbf{n} = \mathbf{r}^{\xi}\times\mathbf{r}^{\zeta}/\sqrt{g}$, i.e., $\mathbf{H}\cdot\mathbf{r}^{\mu} = 0$, $(\mu = \xi, \zeta)$, the term in the first braces { } in (22) vanishes. However, if $R_i = 0$, i.e., the operator-ordering factors f_i are equal to constant, the terms in the second braces { } in (22) have nonzero contribution that is $-H^2$. To see this fact, we need to use the Weingarten formula $\partial_\mu\mathbf{n} \equiv \mathbf{n}_\mu = -b_{\mu\upsilon}\mathbf{r}^{\upsilon}$ [4] and a relation $\mathbf{r}^{\mu}\cdot\partial_\mu\mathbf{n} = -\mathbf{r}^{\mu}\cdot\mathbf{r}^{\upsilon}b_{\mu\upsilon} = -g^{\mu\upsilon}b_{\mu\upsilon} = -2H$ [4]. Then $\mathbf{H}$-dependent term in the second braces { } in (22) is then $x_i^{\mu}(\partial_\mu H_i) + H_i H_i = \mathbf{r}^{\mu}\cdot(\partial_\mu\mathbf{H}) + \mathbf{H}\cdot\mathbf{H} = -H^2$. So, if $f_i =$ const, i.e., $\mathbf{R} = 0$, the result (19) holds. However, the presence of the operator-ordering terms R_i may cancel out the excess terms, making the terms in the second braces { } in (22) vanish. This requirement leads to the following equation in vector form

$$\mathbf{r}^{\mu}\cdot((\partial_\mu\mathbf{H}) - (\partial_\mu\mathbf{R})) + (H_i H_i - R_i R_i) = 0. \tag{23}$$

It is a nonlinear differential equation and trivial case $f_i =$ const, $\mathbf{R} = 0$, can never solve it unless $H = 0$. A particular solution for $\mathbf{R}$ is evidently,

$$\mathbf{R} = \mathbf{H} = H\mathbf{n}. \tag{24}$$

When the motion is constraint-free or in a flat plane, i.e., when $H = 0$, the factors f_i become trivial for $f_i =$ const from equation (21).

4. Other two operator-orderings in a kinetic operator

In our previous concrete approach [7], we use the following form of the kinetic operator:

$$T1 = \frac{1}{2m}\sum_{i=1}^{3}\frac{1}{f_i(x,y,z)}p_i f_i(x,y,z) p_i. \tag{25}$$

It is also tempted to use

$$T2 = \frac{1}{2m}\sum_{i=1}^{3} p_i f_i(x, y, z) p_i \frac{1}{f_i(x, y, z)}. \tag{26}$$

The operator-ordering problem presented in equations (25) and (26) differs from equation (6) only in the way of distribution of the operator-ordering factors $f_i(x, y, z)$. Next, we prove that these factors $f_i(x, y, z)$ have exactly the same form as is given by equation (24).

Expanding the right-hand side of equations (25) and (26), we find respectively

$$\begin{aligned} T1 &= -\frac{\hbar^2}{2m}\left(x_i^{\mu}\partial_{\mu} + H_i + R_i\right)\left(x_i^{\upsilon}\partial_{\upsilon} + H_i\right) \\ &= -\frac{\hbar^2}{2m}\left(x_i^{\mu}\partial_{\mu}x_i^{\upsilon}\partial_{\upsilon} + \left\{(2H_i + R_i)x_i^{\upsilon}\right\}\partial_{\upsilon} + \left\{x_i^{\mu}(\partial_{\mu}H_i) + R_i H_i + H_i H_i\right\}\right), \end{aligned} \tag{27}$$

and

$$\begin{aligned} T2 &= -\frac{\hbar^2}{2m}\left(x_i^{\mu}\partial_{\mu} + H_i\right)\left(x_i^{\upsilon}\partial_{\upsilon} + H_i - R_i\right) \\ &= -\frac{\hbar^2}{2m}\left(x_i^{\mu}\partial_{\mu}x_i^{\upsilon}\partial_{\upsilon} - \left\{R_i x_i^{\mu}\right\}\partial_{\mu} + \left\{x_i^{\mu}((\partial_{\mu}H_i) - (\partial_{\mu}R_i)) + H_i(H_i - R_i)\right\}\right). \end{aligned} \tag{28}$$

This requirement that the terms in two braces { } in (27) vanish simultaneously leads to a set of two equations:

$$\begin{cases} \mathbf{R}\cdot\mathbf{r}^{\mu} = 0, & (\mu = \xi, \zeta) \\ -H^2 + \mathbf{R}\cdot\mathbf{H} = 0. \end{cases} \tag{29}$$

The same requirement for (28) leads to another set of two equations:

$$\begin{cases} \mathbf{R}\cdot\mathbf{r}^{\mu} = 0, & (\mu = \xi, \zeta) \\ \mathbf{r}^{\mu}\cdot(\partial_{\mu}\mathbf{H}) - (\partial_{\mu}\mathbf{R}) + \mathbf{H}\cdot(\mathbf{H} - \mathbf{R}) = 0. \end{cases} \tag{30}$$

The first equation in either set (29) or (30) $\mathbf{R}\cdot\mathbf{r}^{\mu} = 0$ states nothing but a fact that the direction of $\mathbf{R}$ is along the normal $\mathbf{n}$. The second equation in either set (29) or (30) determines the magnitude of $\mathbf{R}$, and the unique solution is $R = H$.

5. Examples

In this section, two ideal quantum dots, the spheroidal surface [2] and the toroidal surface [3] will be utilized to illustrate the abstract results developed above.

5.1. Operators on the spheroidal surface

The spheroidal surface is with two local coordinates $\theta \in [0, 2\pi)$, $\varphi \in [0, 2\pi)$,

$$\mathbf{r} = (x, y, z) = (a\sin\theta\cos\varphi, a\sin\theta\sin\varphi, b\cos\theta), \tag{31}$$

where a and b denote two distinct axes. The convariant derivatives $\mathbf{r}_{\mu}$ and contravariant derivatives $\mathbf{r}^{\mu}$ can be easily computed and the results are respectively

$$\begin{pmatrix}\mathbf{r}_{\theta}\\ \mathbf{r}_{\varphi}\end{pmatrix} = \begin{pmatrix} a\cos\theta\cos\varphi, & a\cos\theta\sin\varphi, & -b\sin\theta \\ -a\sin\theta\sin\varphi, & a\sin\theta\cos\varphi, & 0 \end{pmatrix}, \tag{32}$$

$$\begin{pmatrix} \mathbf{r}^\theta \equiv g^{\theta\upsilon}\mathbf{r}_\upsilon \\ \mathbf{r}^\varphi \equiv g^{\varphi\upsilon}\mathbf{r}_\upsilon \end{pmatrix} = \frac{1}{a}\begin{pmatrix} G(a,b,\theta)\cos\theta\cos\varphi, & G(a,b,\theta)\cos\theta\sin\varphi, & -G(a,b,\theta)b/a\sin\theta \\ -\csc\theta\sin\varphi, & \csc\theta\cos\varphi, & 0 \end{pmatrix}, \tag{33}$$

where $G(a, b, \theta) = 2/(1 + \varepsilon^2 + (1 - \varepsilon^2)\cos 2\theta)$ with $\varepsilon = b/a$. The normal $\mathbf{n}$ and the mean curvature H are given by respectively

$$\mathbf{n} = \sqrt{G(a,b,\theta)}(\varepsilon\sin\theta\cos\varphi, \varepsilon\sin\theta\sin\varphi, \cos\theta), \tag{34}$$

$$H(a,b,\theta) = -b/(4a^2)(3 + \varepsilon^2 + (1-\varepsilon^2)\cos 2\theta)G(a,b,\theta)^{3/2}. \tag{35}$$

The Hermitian Cartesian momentum operators p_i $(i = 1, 2, 3)$ are

$$p_x = -\mathrm{i}\hbar\frac{1}{a}\left(\cos\theta\cos\varphi G(a,b,\theta)\frac{\partial}{\partial\theta} - \csc\theta\sin\varphi\frac{\partial}{\partial\varphi} - F(a,b,\theta)\cos\varphi\sin\theta\right), \tag{36}$$

$$p_y = -\mathrm{i}\hbar\frac{1}{a}\left(\cos\theta\sin\varphi G(a,b,\theta)\frac{\partial}{\partial\theta} + \cos\varphi\csc\theta\frac{\partial}{\partial\varphi} - F(a,b,\theta)\sin\theta\sin\varphi\right), \tag{37}$$

$$p_z = \mathrm{i}\hbar\left(\frac{b}{a^2}\sin\theta G(a,b,\theta)\frac{\partial}{\partial\theta} + \frac{1}{b}F(a,b,\theta)\cos\theta\right), \tag{38}$$

where $F(a, b, \theta) = \varepsilon^2(3 + \varepsilon^2 + (1 - \varepsilon^2)\cos 2\theta)G(a, b, \theta)^2/4$. The factor functions ($f_x$, f_y, f_z) determined by equation $R_i = Hn_i$ (21) have special solutions:

$$f_x = G(a,b,\theta)^{1/4}(\cos\theta)^{\frac{a^2+b^2}{2a^2}}, \tag{39}$$

$$f_y = G(a,b,\theta)^{1/4}(\cos\theta)^{\frac{a^2+b^2}{2a^2}}, \tag{40}$$

$$f_z = G(a,b,\theta)^{1/4}\sin\theta. \tag{41}$$

When the spheroid becomes a sphere with $a = b$, we have $\varepsilon = 1, G(a, b, \theta) = 1$, $F(a, b, \theta) = 1$ and $H(a, b, \theta) = -1$. All results above readily reduce to those for sphere [7].

5.2. Operators on the toroidal surface

The toroidal surface is with two local coordinates $\theta \in [0, 2\pi)$, $\varphi \in [0, 2\pi)$,

$$\mathbf{r} = ((a + b\sin\theta)\cos\varphi, (a + b\sin\theta)\sin\varphi, b\cos\theta), \qquad (a > b)$$

where a and b denote two distinct radii. The convariant derivatives $\mathbf{r}_\mu$ and contravariant derivatives $\mathbf{r}^\mu$ can be easily computed and the results are respectively

$$\begin{pmatrix} \mathbf{r}_\theta \\ \mathbf{r}_\varphi \end{pmatrix} = \begin{pmatrix} b\cos\theta\cos\varphi, & b\cos\theta\sin\varphi, & -b\sin\theta \\ -(a+b\sin\theta)\sin\varphi, & (a+b\sin\theta)\cos\varphi, & 0 \end{pmatrix}, \tag{42}$$

$$\begin{pmatrix} \mathbf{r}^\theta \equiv g^{\theta\upsilon}\mathbf{r}_\upsilon \\ \mathbf{r}^\varphi \equiv g^{\varphi\upsilon}\mathbf{r}_\upsilon \end{pmatrix} = \begin{pmatrix} \frac{\cos\theta\cos\varphi}{b}, & \frac{\cos\theta\sin\varphi}{b}, & -\frac{\sin\theta}{b} \\ -\frac{\sin\varphi}{a+b\sin\theta}, & \frac{\cos\varphi}{a+b\sin\theta}, & 0 \end{pmatrix}. \tag{43}$$

The normal $\mathbf{n}$ and the mean curvature H are given by respectively

$$\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \tag{44}$$

$$H = -\frac{a + 2b\sin\theta}{2b(a + b\sin\theta)}. \tag{45}$$

With use of the above expression for mean curvature H (45), the Hermitian Cartesian momentum operators p_i $(i = 1, 2, 3)$ are given by

$$p_x = -\mathrm{i}\hbar\left(\frac{\cos\theta\cos\varphi}{b}\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{a + b\sin\theta}\frac{\partial}{\partial\varphi} + H\sin\theta\cos\varphi\right), \tag{46}$$

$$p_y = -\mathrm{i}\hbar\left(\frac{\cos\theta\sin\varphi}{b}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{a + b\sin\theta}\frac{\partial}{\partial\varphi} + H\sin\theta\sin\varphi\right), \tag{47}$$

$$p_z = \mathrm{i}\hbar\left(\frac{\sin\theta}{b}\frac{\partial}{\partial\theta} - H\cos\theta\right). \tag{48}$$

The factor functions (f_x, f_y, f_z) determined by equation $R_i = Hn_i$ (21) have special solutions:

$$f_x = (a + b\sin\theta)^{\frac{1}{2}\frac{a^2}{a^2-b^2}}(1 + \sin\theta)^{\frac{1}{4}\frac{a-2b}{a-b}}(\sin\theta - 1)^{\frac{1}{4}\frac{a+2b}{a+b}}, \tag{49}$$

$$f_y = (a + b\sin\theta)^{\frac{1}{2}\frac{a^2}{a^2-b^2}}(1 + \sin\theta)^{\frac{1}{4}\frac{a-2b}{a-b}}(\sin\theta - 1)^{\frac{1}{4}\frac{a+2b}{a+b}}, \tag{50}$$

$$f_z = \sqrt{(a + b\sin\theta)\sin\theta}. \tag{51}$$

In an extreme case $a = 0$, the torus becomes a sphere of radius b, and all results above also reduce to those for sphere [7].

6. Remarks and summary

In classical mechanics for a particle moving on the curved surface M embedded in R^3, the local curved coordinates (ξ, ζ) on M and the Cartesian coordinates (x, y, z) in R^3 seem to play equal roles in the description of its classical motion, for the results written in these two coordinate systems are related to each other by coordinate transformation. On the other hand, in light of the canonical variable, neither the Cartesian coordinates nor the Cartesian momentum can be taken as canonical variables. Any pair of Cartesian variables (x_i, p_i) is no longer canonical conjugate to each other. Even looking for the canonically conjugate variables for these Cartesian variables x_i, p_i does not seem to be a physically meaningful task. In contrast, since the variables canonically conjugate to the local coordinate variables (ξ, ζ) naturally exist, the quantization based on the conjugate variables can be easily preformed with the help of the so-called canonical quantization rules. However, though so far quantum mechanics uses the local coordinate system only, it contains nice results associated with the Cartesian coordinates.

The present work shows a compact and abstract result for Hermitian Cartesian momentum operators describing the particle moving on the curved surface M embedded in R^3, and it is a constant factor $-\mathrm{i}\hbar$ times the mean curvature vector field $H\mathbf{n}$ added to the usual differential $\mathbf{r}^\mu\partial_\mu$. With use of this Cartesian momentum, the same operator-ordering factors can be distributed in three different ways and all lead to the correct quantum kinetic energy. These operator-ordering factors become dummy in classical limit and reduce to be constant for the motion is constraint-free or in the flat plane. Thus, the present study demonstrates that the Cartesian coordinates is also useful in quantum mechanics and casts a new insight into the understanding of the classical correspondence of quantum mechanics [8–10].

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References

[1] Kleinert H 1990 *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics* (Singapore: World Scientific) pp 34–7, 451–3
[2] Cantele G, Ninno D and Iadonisi G 2000 *Phys. Rev.* B **61** 13730
[3] Encinosa M and Mott L 2003 *Phys. Rev.* A **68** 014102
[4] do Carmo M P 1976 *Differential Geometry of Curves and Surfaces* (Englewood Cliffs, NJ: Prentice-Hall)
[5] Bohm D 1951 *Quantum Theory* (Englewood Cliffs, NJ: Prentice-Hall)
[6] Ou-Yang Z C, Liu J X and Xie Y Z 1999 *Geometrical Methods in the Elastic Theory of Membranes in Liquid Crystal Phases* (Singapore: World Scientific) p 51
[7] Liu Q H and Liu T G 2003 *Int. Theor. Phys.* **42** 2877
Liu Q H, Hou J X, Xiao Y P and Li L X 2004 *Int. Theor. Phys.* **43** 1011
Xiao Y P, Lai M M, Hou J X, Chen X W and Liu Q H 2005 *Commun. Theor. Phys.* **44** 49
Liu Q H 2006 *Int. J. Theor. Phys.* **45** 2167
Lai M M, Wang X, Xiao Y P and Liu Q H 2005 *Commun. Theor. Phys.* **46** 843
Wang X, Xiao Y P, Liu T G, Lai M M and Rao J 2006 *Int. Theor. Phys.* **45** 2509
[8] Liu Q H 1999 *J. Phys. A: Math. Gen.* **32** L57
Liu Q H *et al* 2002 *J. Math. Phys.* **43** 170
[9] Liu Q H and Hu B 2001 *J. Phys. A: Math. Gen.* **34** 5713
[10] Liu Q H, Wang X and Yu Y B 2007 *Int. J. Theor. Phys.* **46** 424–9